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a continuous denotational semantics

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Abstract

We present a denotational semantics based on Banach spaces; it is inspired from the familiar coherent semantics of linear logic, the role of coherence being played by the norm: coherence is rendered by a supremum, whereas incoherence is rendered by a sum, and cliques are rendered by vectors of norm at most 1. The basic constructs of linear (and therefore intuitionistic) logic are implemented in this framework: positive connectives yield l^1-like norms and negative connectives yield l^∞-like norms. The problem of non-reflexivity of Banach spaces is handled by "specifying the dual in advance", whereas the exponential connectives (i.e. intuitionistic implication) are handled by means of analytical functions on the open unit ball. The fact that this ball is open (and not closed) explains the absence of a simple solution to the question of a topological cartesian closed category: our analytical maps send an open ball into a closed one and therefore do not compose. However, a slight modification of the logical system allowing to multiply a function by a scalar of modulus <1 is enough to cope with this problem. The logical status of the new system should be clarified.

1. Introduction

We shall not discuss the general issue of topology and logic (e.g. logical approach to topology as in – say – formal topologies), but the restricted question of adding topological features to logic.

1.1. Topology in logic

1.1.1. Scott domains

Logic is by nature discrete; in many situations we would like to connect its rules with analysis, i.e. with real or complex numbers. Naïve attempts at introducing some

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"fuzziness" in logic eventually ended in fuzzy... methodology and notorious parascience. The most important attempt at reconciling continuity and logic amounts to the works of Dana Scott (and independently Ershov), around 1970, see e.g. [10]. The problem at stake was to give a concrete model of the Heyting–Kolmogoroff paradigm of "proofs as functions", in which each logical formula is interpreted by a set, and logical implication $A \Rightarrow B$ is the set of functions from A to B. The set-theoretic interpretation is too brutal in view of the constructive character of this "semantics of proofs": the proposal was therefore to replace sets with topological spaces and therefore functions with continuous ones. This was not an easy endeavor, since the function space has to be given in turn a topology... and two major possibilities appear, namely pointwise and uniform convergence. For instance take $A = B = [0,1]$: the continuous interpretation of $A, A \Rightarrow B \vdash B$, i.e. of the functional $x, f \rightsquigarrow f(x)$ requires uniform convergence; but the interpretation of $A \vdash (A \Rightarrow B) \Rightarrow B$, i.e. of the functional $x \rightsquigarrow ev_x$, where ev_x is the evaluation $ev_x(f) = f(x)$ is discontinuous if we equip $(A \Rightarrow B) \Rightarrow B$ with uniform convergence. The solution found by Scott was to avoid the dichotomy "pointwise vs. uniform" by means of a restriction to certain non-uniformizable spaces. The problem is that these spaces are far astray from standard topology [1] (e.g. $\mathbb{R}$, $\mathbb{C}$); indeed they are not even Hausdorff. By the way Scott domains can be described in terms of algebraic complete partial orders (c.p.o.) and continuous monotone maps, and it seems that this alternative presentation corresponds to the true spirit of the construction. Anyway, in spite of its limited topological aspects, Scott and Ershov initiated *denotational semantics*, which is the model-theory of proofs, and more recently of computations.

1.1.2. Compactly generated spaces

The problem solved by Scott was the construction of a closed cartesian category made of topological spaces, a problem independently addressed by category theorists, namely the construction of a cartesian closed category: in such a category, one can construct products and function spaces so as to get a canonical isomorphism $Mor(X \times Y, Z) \simeq Mor(X, Z^Y)$. There is indeed another topology on the function space, the *compact-open topology*, which works for a special kind of Hausdorff spaces, namely *compactly generated spaces* invented by Kelley in 1955; unfortunately, these spaces are not naturally closed under products and function spaces, and the product and compact-open topologies must be modified ("Kelleyfied" see [9]) in order to get the right objects. The weak point of this approach is that the categorical product is not the topological product; this is perhaps why the only living tradition of continuous semantics is the one of Scott–Ershov.

[1] In Scott domains separate continuity implies continuity, which sounds rather strange from the topological standpoint.

1.1.3. Quantitative and qualitative domains

My first work in denotational semantics [5] (1984) was based on the a priori that Scott semantics had nothing or little to do with topology. The idea was to revisit the order-theoretic approach in the light of category theory: if an order relation is seen as a (degenerated) category, then a monotone map is a functor and continuity is preservation of direct limits; furthermore this viewpoint suggests additional preservations, with no "topological" counterparts, such as pull-backs or kernels. The result of these investigations was a pair of semantics:

- *Quantitative semantics* was based on the idea of counting basic data with multiplicities, i.e. to work with multisets of basic tokens; functions were indeed definable by means of formal power series, with – in the good cases – integer coefficients; no real topology was involved, since in "bad" cases these coefficients could become infinite …
- *Qualitative semantics* was a simplification of quantitative semantics, neglecting multiplicities, but replacing it with a notion of compatibility between tokens; however something of the multiplicities was still present, in terms of *stability*: $a \cup b \sqsubset X \Rightarrow F(a \cap b) = F(a) \cap F(b)$, which is a pull-back condition.

Quantitative semantics had a very marginal publicity, but was responsible for the discovery of linearity (i.e. the case when the power series is of degree 1). Linearity was eventually developed in the framework of *coherent spaces*, a simplification of qualitative domains, with binary compatibility.

1.1.4. Coherent spaces

A *coherent space* (see e.g. [7]) is a graph X (i.e. a set and a *coherence* relation), and we are interested in the cliques $a \sqsubset X$ of X, i.e. in sets of pairwise compatible points of our graph. A linear map from X to Y is just a map from cliques to cliques which preserves arbitrary sums of cliques: by sum I mean a union of disjoint cliques, provided it is still a clique. We see that this definition (which is the ultimate simplification of Scott's definition) has very little topology in it (infinite unions), and is slightly more algebraic, although the impossibility of forming something like $-a$ (the opposite of a clique a) is a severe limitation.

Nevertheless, linear logic was built around this basic semantics, with three layers of connectives, *multiplicatives* (tensor product and cotensor product), *additives* (direct sum and product) and *exponentials* (comonoid and cocomonoid), with the same brute expressive power as the usual (intuitionistic) logic modelized by Scott, but more subtlety, in particular the presence of an involutive negation $X^{\perp}$, which is basically the complementary graph.

1.1.5. Vector spaces

Linear negation is clearly analogous to the formation of the dual space in algebra. Indeed if we leave aside the exponential connectives, the rules of linear logic can be modelized in finite-dimensional vector spaces … maybe too easily, since the tensor and the cotensor are identified, and sum and product as well. In infinite dimension the two multiplicatives are distinct, but the spaces are no longer equal to their second dual; this is why Blute and Philip Scott in their paper [2] used an old trick of Lefschetz to cope with infinite dimension, namely to introduce a topology to cut the size of the dual, so as to preserve involutivity. Again this topological trick belongs more to the spirit of algebra than to the spirit of topology.

The paper [2] basically deals with multiplicatives; in order to separate the two additives the authors realized (work in progress, see the forthcoming [3]) that normed spaces can do it, e.g. using the distinction l^∞/l^1, which is consistent with the very contents of our paper.

1.2. Coherent Banach spaces

1.2.1. About the norm

The idea is to give a continuous version of coherent spaces; the experience of linear logic tells us that we must seek a vector space. A topological space must therefore be considered, and among such spaces, Banach Spaces are the most natural ones. More precisely normed space are the simplest examples of topological vector spaces; the completeness of the space is clearly needed in order to mimic infinite sums of cliques... finally these spaces will turn out to be complex ones, in order to apply the machinery of complex analysis. The norm defines a topology, but it makes sense in itself: in finite dimension all norms are equivalent, but we must distinguish between two spaces of the same finite dimension. OK, but then what is the actual meaning of the norm? In coherent spaces we had points and sets, some of these sets being cliques; here we have only vectors. Our claim is that the norm serves to distinguish between "cliques" and "non-cliques". Concretely, the statement $\|x\| \leqslant 1$ is the analogue of "a is a clique". The idea works wonderfully: in coherent spaces the two additive connectives differ because a clique in $X \,\&\, Y$ is the disjoint sum of a clique in X and a clique in Y, whereas a clique in $X \oplus Y$ is either a clique in X or a clique in Y. Here we can equip the direct sum of Banach spaces E, F with two norms, the supremum (l^∞-norm), and the sum (l^1-norm): in the first case $e \oplus f$ will receive the norm $\sup(\|e\|, \|f\|)$, and a direct sum of "cliques" will remain a "clique", whereas in the second case, the norm $\|e\| + \|f\|$ induces an incompatibility between "cliques", which might go as far as mutual exclusion, e.g. if $\|e\| = 1$, with the additional possibility to pass continuously from one side to another.

1.2.2. About negation

Linear negation is involutive, whereas Banach spaces are in general not reflexive. For instance $c_0' = l^1$, $l^{1\prime} = l^\infty$, $l^{\infty\prime} \supsetneq l^1$, etc. shows that not only a space may be distinct from a second dual, but that a dual can be distinct from a third dual. Of course certain very good spaces are reflexive, typically the Hilbert space l^2; but l^∞/l^1-norms fit so well the additive case... that we must quit the Hilbertian paradise. There is a solution, namely to give the dual in advance. [2] This means that we are given a pair of spaces $E, E^{\perp}$, each of them being a subspace of the dual of the other. This can be said in a more abstract way, by introducing a bilinear form between the two spaces and requiring a certain adequation between the norms and the bilinear form. The resulting objects are called *coherent Banach spaces*, or CBS.

1.2.3. About multiplicatives

The first thing is to get a decent tensor and a decent cotensor. Modulo dualization, this can be extracted from an appropriate notion of morphism between CBS E and F: a morphism will be a bounded linear map φ from E to F, which induces (as usual) an adjoint map φ' from F' to E'; now remember that $E^{\perp} \subset E'$, $F^{\perp} \subset F'$: we require that φ' actually maps $F^{\perp}$ into $E^{\perp}$. In order to state the properties of the tensor product a (straightforward) multilinear variant of the same notion has to be introduced. Observe that the norm of the cotensor is of the style l^{∞} (a supremum), whereas the norm of the tensor is of style l^1. In general the positive operations $(\otimes, \oplus, !)$ involve l^1-norms, whereas negative ones $(\invamp, \ \&\ , ?)$ involve l^{∞}-norms.

1.2.4. Exponentials

Much more delicate is the case of exponentials. These connectives arise from the need to cope with the want of linearity, in analogy to the symmetric tensor algebra. The experience on quantitative semantics suggests to take analytical functions defined by power series; the coefficients lay in some symmetrical cotensor power of the space. Typically the space? $E^{\perp}$ consists of functions φ defined on E by means of power series $\varphi(e) = \sum \varphi_n(\otimes^n e)$. The only delicate question is the choice of the domain $\mathscr{D}$ of definition of φ. Here we have to remember the essential isomorphism that exponentials must satisfy, namely $!(E \ \&\ F) \simeq !E \otimes !F$, i.e. that ! transforms the additive conjunction into the multiplicative conjunction, and which is nothing but a simplified form of the basic isomorphism of cartesian closed categories $Mor(X \times Y, Z) \simeq Mor(X, Z^Y)$. In terms of functions, this means that an analytical function defined on $E \ \&\ F$ can be identified with an analytical function sending an element of E to an analytical function defined on F … & involves a l^{∞}-norm, hence the only possible norm for analytical functions is also a l^{∞}-norm (our isomorphism must be isometric), i.e. $\|\varphi\| = \sup\{|\varphi(e)|; e \in \mathscr{D}\}$, and by Liouville's theorem, this supremum is likely to be infinite if $\mathscr{D} = E$. From this it follows that $\mathscr{D}$ is a ball, and only the unit ball makes sense. Now it remains to check whether or not this ball is open or closed: but requiring that our function extends continuously to the closed ball is unmanageable, see below. Our functions are therefore defined on open unit balls. The usual machinery of analytical functions, Cauchy integral, geometric series,… works as expected: this is why our spaces are complex.

[2] A tradition amounting to Mackey, Barr, Chu… see for instance [1].

$!E$ is generated by the evaluations $!e$ defined by $\langle !e, \varphi \rangle = \varphi(e)$ for $\|e\| < 1$ (and for instance contains the Cauchy integrals, which are limits of barycenters of evaluations); but when $\|e_n\|, \|f_n\|$ tends to 1, with $e_n \neq f_n$, the norm of $!e - !f$ tends to 2. This shows that there is a problem at the border: if we try to work with the closed ball, then the points $!e$ would be at pairwise distance 2 when $\|e\| = 1$, contradicting the expected continuity of the map $x \rightsquigarrow !x$. By the way we are doing nothing but rediscovering the impossibility of handling evaluation on the basis of uniform continuity.

A way to synthesize the properties of our exponentials would be to establish a universal property. We indeed propose two solutions (comonoid, strong comonoid) but there is always a small mismatch, which by the way corresponds to the problem we met at the border of the ball. The category-theoretic status of exponentials is still in want of a clarification.

1.2.5. Coefficients

There is therefore a problem with the interpretation, which is perhaps also its main quality: the basic logical constructions have norm 1, hence our basic analytical functions will have norm 1 too, which means that they send an open ball into a closed ball… and therefore composition of analytical maps is impossible ! We spent a long time on this problem, to finally reach the following conclusion: let us allow in proofs the plugging of complex parameters of modulus < 1; then when an object should be in the open ball, simply slightly shrink it by multiplication with an adequate scalar. This induces a modification of the rules of existing logical systems, but all essential properties are preserved; this is the weighted calculus that we present here. [3]

1.3. Open questions

1.3.1. Extension to second order

A first question is to determine to which extent our spaces remain “small”, let us say of the power of the continuum; remember that Scott semantics, coherent spaces, etc. remain small enough; typically all useful coherent spaces are denumerable, hence have a continuum of cliques. The answer could be in the building of a separable predual for each of our spaces, but this is not obvious. A neighboring problem is that of the extension to second-order, i.e. *parametricity*. In coherent spaces, every space can be approximated by finite ones, and parametricity could be defined via a commutation to these approximations. Here we meet the problem that our constructions do not obviously commute with approximations (which is connected to smallness) and the fact that Banach spaces cannot be approximated by finite-dimensional ones.

1.3.2. Proof-nets

If coefficients and the corresponding rules actually make sense, then it will be necessary to develop a clean syntax. So what about proof-nets in this enlarged context?

[3] Rather a first draft: many variants of the same calculus are possible.

1.3.3. So what?

As far as continuous semantics is concerned, it is obvious that our solution is clean and satisfactory, even if we are still in want of an extension to second order. But we are not producing semantics for “l’Art pour l’Art”, and there should be a feedback. I can foresee certain applications:

- The existence of a continuous semantics should be exploited to prove technical results about usual (finite) syntax.
- The complex parameters that occur in the rule “scalar” are surely not mere technicalities; what do they mean, how can they be used? Can we connect this with some probabilistic intuitions concerning non-determinism? This has to be related with completeness issues, i.e. to which extent can we formulate a *denotational completeness* theorem w.r.t. our semantics: our recent paper [8] presents a general framework which yields completeness (i.e. the statement that only logical operations can be implemented, which requires some restriction on the shape of implementations), essentially by replacing spaces by “free modules over a comonoid”, and this should adapt, *mutatis mutandis* to our new framework... but keep in mind that what is important in a completeness theorem is that the restrictions on the shape of implementations should be non-contrived.
- One of the immediate outputs of coherent spaces was to individuate new connectives; something similar happens here, typically at the additive level, where l^p/l^q can be used instead of l^∞/l^1. This induces new “connectives”, which are not linked to any existing logical practice – unlike the linear connectives which legalized underground operations – . The question of giving a sense to these connectives might be of great interest. However our attempts at giving a sequent calculus for these connectives (e.g. the self-dual connective corresponding to l^2) are not convincing enough: not enough “nice” properties are preserved. Of course they might satisfy alternative properties, but not enough practice has been accumulated to find which ones should be considered, anyway these “connectives” are tantalizing.

2. Multiplicative and Additive constructions

2.1. Coherent Banach spaces

Definition 1. A Coherent Banach space (CBS) consists of the following data:

(i) complex Banach spaces $E, E^\perp$,

(ii) a bilinear form $\langle .,. \rangle$ from $E, E^\perp$ to $\mathbb{C}$ enjoying

$$\forall x \in E \quad \|x\| = \sup\{|\langle x, y\rangle|;\ y \in E^\perp, \|y\| \leqslant 1\}, \tag{1}$$

$$\forall y \in E^\perp \quad \|y\| = \sup\{|\langle x, y\rangle|;\ x \in E, \|x\| \leqslant 1\}. \tag{2}$$

In other terms, each of the two spaces $E, E^{\perp}$ can be identified with a subspace of the dual of the other:

$$E \hookrightarrow E^{\perp'}, \qquad E^{\perp} \hookrightarrow E'.$$

The typical example of a CBS is $(E, E', \langle .,. \rangle)$ with $\langle e, e' \rangle = e'(e)$; condition (CBS2) is nothing but the definition of the norm of E' whereas (CBS1) follows from the Hahn–Banach theorem. A typical abuse of notation will be to refer to a CBS by naming it E, thus considering that $E^{\perp}$ and $\langle .,. \rangle$ are clear from the context. In fact the basic example that we can keep in mind is that of the pair $(l^{\infty}, l^1, \langle .,. \rangle)$, which is the analogue of the flat coherent space: $i \asymp j$ for $i, j \in \mathbb{N}$.

Definition 2. The linear negation of $(E, E^{\perp}, \langle .,. \rangle)$ is defined as $(E^{\perp}, E, \widetilde{\langle .,. \rangle})$, with $\widetilde{\langle e', e \rangle} = \langle e, e' \rangle$.

Linear negation is clearly involutive.

2.2. CBS as a multicategory

Definition 3. A *coherent multilinear form* μ on $E_1, \ldots, E_n$ is a bounded multilinear form on $E_1, \ldots, E_n$ such that, for $i = 1, \ldots, n$:

$$\forall e_1 \in E_1 \ldots \widehat{\forall e_i \in E_i} \ldots \forall e_n \in E_n, \quad \exists e_i' \in E_i^{\perp} \forall e_i \in E_i, \quad \mu(e_1, \ldots, e_n) = \langle e_i, e_i' \rangle$$

As usual $\widehat{\forall e_i \in E_i}$ stands for a missing item.

In other terms for $i \leqslant n$ we require that the canonical map μ_i from $E_1 \times \cdots \times \hat{E}_i \times \cdots \times E_n$ into E_i' induced by μ is actually into $E_i^{\perp}$.

Definition 4. A *coherent n-morphism* $\varphi \in Hom_n(E_1, \ldots, E_n; F)$ is a multilinear map such that $\mu(e_1, \ldots, e_n, f') = \langle \varphi(e_1, \ldots, e_n), f' \rangle$ defines a coherent form on $E_1, \ldots, E_n, F^{\perp}$.

A coherent n-morphism is therefore nothing but one of the $n+1$ maps μ_i associated with a coherent $n+1$-morphism. A coherent n-morphism can be attributed the norm $\|\varphi\| = sup\{|\varphi(e_1, \ldots, e_n)|; \|e_1\|, \ldots, \|e_n\| \leqslant 1\}$, which is the same (if $\varphi = \mu_{n+1}$) as $\|\mu\| = sup\{|\mu(e_1, \ldots, e_n, f')|; \|e_1\|, \ldots, \|e_n\|, \|f'\| \leqslant 1\}$. Particular cases are:

- If $n = 0$, a 0-morphism $\varphi \in Hom_0(; F)$ is nothing but a point of F.
- The crucial case is $n = 1$; a 1-morphism (simply: morphism) from E to F (notation: $\varphi \in Mor(E, F)$) is a bounded linear map from E into F such that there is a map $\varphi^{\perp}$ from $F^{\perp}$ into $E^{\perp}$ such that

$$\forall e \in E \quad \forall f' \in F^{\perp}, \quad \langle \varphi(e), f' \rangle = \langle e, \varphi^{\perp}(e') \rangle.$$

Coherent n-morphisms can be composed in an obvious way:
if $\varphi \in Hom_n(E_1, \ldots, E_n; F_{m+1})$ and $\psi \in Hom_{m+1}(F_1, \ldots, F_{m+1}; G)$ then one can define a $n+m$-morphism $\iota = \psi\varphi \in Hom_{n+m}(E_1, \ldots, E_n, F_1, \ldots, F_m; G)$ by $\iota(e_1, \ldots, e_n, f_1, \ldots, f_m)$

$=\psi(f_1,\dots,f_m,\varphi(e_1,\dots,e_n))$. Composition is associative, and the identity maps $id_E \in Mor(E,E)$ are neutral. Moreover, the multicategory is symmetrical (and this is why we didn't bother with defining composition of φ,ψ when the target of φ is any of the F_i): given a permutation σ of $\{1,\dots,n\}$ and $\varphi \in Hom_n(E_1,\dots,E_n;F)$, we can define $\sigma(\varphi) \in Hom_n(E_{\sigma(1)},\dots,E_{\sigma(n)};F)$, by $\sigma(\varphi)(e_1,\dots,e_n) = \varphi(e_{\sigma(1)},\dots,e_{\sigma(n)})$.

2.3. Multiplicatives

A multicategory is the right place where a tensor product might be defined; indeed the tensor product is the solution to a universal problem, namely:

Theorem 1. *Given E,F, one can find a CBS $E\otimes F$ together with a 2-morphism $\varphi \in Hom_2(E,F;E\otimes F)$ inducing a bijection between $Hom_{n+2}(\Gamma,E,F;G)$ and $Hom_{n+1}(\Gamma,E\otimes F;G)$ for any sequence Γ,G of CBS.*

Proof. The proof basically consists of the following definition:

Definition 5. The CBS

$$(E\otimes F, E^{\perp}\mathbin{⅋}F^{\perp}, \langle .,. \rangle)$$

is defined as follows:

- $E^{\perp}\mathbin{⅋}F^{\perp}$ consists of all coherent bilinear forms b on E,F, equipped with the obvious norm

$$\|b\| = \sup\{|b(x,y)|; \|x\|,\|y\| \leqslant 1\}.$$

- Consider the algebraic tensor product $E\odot F$; any $b \in E^{\perp}\mathbin{⅋}F^{\perp}$ induces a linear map from $E\odot F$ to $\mathbb{C}$ and we can therefore equip $E\odot F$ with the semi-norm (indeed a norm by standard algebra)

$$\|a\| = \sup\{|b(a)|; b \in E^{*}\mathbin{⅋}F^{*}, \|b\| \leqslant 1\}.$$

  We define $E\otimes F$ to be the completion of $E\odot F$.
- The bilinear form $\langle .,. \rangle$ is obtained by extending the map $b,a \rightsquigarrow b(a)$ to a continuous bilinear map from $E\otimes F, E^{\perp}\mathbin{⅋}F^{\perp}$ to $\mathbb{C}$.

$E\otimes F$ is clearly defined as a subspace of $(E^{\perp}\mathbin{⅋}F^{\perp})'$, which accounts for (CBS1). If $e' \in E^{\perp}, f' \in F^{\perp}$, then $(e'\mathbin{⅋}f')(e,f) = e'(e).f'(f)$ defines an element of $E^{\perp}\mathbin{⅋}F^{\perp}$ and $\|e'\mathbin{⅋}f'\| = \|e'\|.\|f'\|$; from this $\|e\otimes f\| = \|e\|.\|f\|$ for $e\in E, f\in F$ and (CBS2) easily follows. We define the bilinear map φ by $\varphi(e,f) = e\otimes f$, and it is immediate that $\varphi \in Hom_2(E,F;E\otimes F)$. The verification of the universal property is more or less trivial. $\square$

By the way observe that, simultaneously to the tensor product "Times", a cotensor "Par" is defined: with abusive notations $E\mathbin{⅋}F = (E^{\perp}\otimes F^{\perp})^{\perp}$. We introduce the notation $\multimap$ as a shorthand: $E\multimap F = E^{\perp}\mathbin{⅋}F$.

The CBS $\mathbb{C}$, i.e. $(\mathbb{C}, \mathbb{C}, .)$ is neutral w.r.t. the tensor product; since $\mathbb{C}^\perp = \mathbb{C}$, $\mathbb{C}$ is neutral w.r.t. the cotensor product as well.

2.4. Additives

A category is the right place where sums and products might be defined; indeed the sum is introduced as the solution to a familiar universal problem. Here we give its multicategorical version which entails the distributivity of $\otimes$ over $\oplus$.

Theorem 2. *Given E, F, one can find a CBS $E \oplus F$ together with 1-morphisms $\iota_l \in Mor(E, E \oplus F)$, $\iota_r \in Mor(F, E \oplus F)$ inducing a bijection between $Hom_{n+1}(\Gamma, E; G) \times Hom_{n+1}(\Gamma, F; G)$ and $Hom_{n+1}(\Gamma, E \oplus F; G)$ for any sequence Γ, G of CBS.*

Proof. The proof basically consists of the following definition:

Definition 6. The CBS

$$(E \oplus F, E^\perp \& F^\perp, \langle , \rangle)$$

is defined as follows:
- $E \oplus F$ is the direct sum of E, F, equipped with the l^1-norm:

$$\|e \oplus f\| = \|e\| + \|f\|$$

- $E^\perp \& F^\perp$ is the direct sum of $E^\perp, F^\perp$, equipped with the l^∞-norm:

$$\|e' \& f'\| = \sup(\|e'\|, \|f'\|)$$

- $\langle e \oplus f, e' \& f' \rangle = \langle e, e' \rangle + \langle f, f' \rangle$

The familiar duality l^1/l^∞ is used to check (CBS1) and (CBS2). All verifications are trivial. $\square$

Besides the categorical sum "Plus", we simultaneously defined, using the abusive formula $E \& F = (E^\perp \oplus F^\perp)^\perp$, a categorical product "With", with dual properties, e.g. $⅋$ distributes over &.

The CBS0 consisting of the null space (which is therefore self-dual) is neutral w.r.t. sum and product.

3. Exponential constructions

3.1. Scherzo: symmetric tensor powers

The n-ary tensor powers $\otimes^n E$ and $⅋^n E^\perp$ are naturally equipped with an action of the symmetric group. Let $T^n E$ and $S^n E^\perp$ be the respective symmetric subspaces. The projections t, s from the full spaces to their respective symmetric subspaces are both defined as a barycenter $(1/n!)\sum \sigma(x)$, and have therefore norm $\leqslant 1$; since $\langle e, s(e')\rangle = \langle t(e), e'\rangle$ it follows that $(T^n E, S^n E^\perp, \langle .,.\rangle)$, with induced norm and bilinear form is still a CBS. But $S^n E^\perp$ bears another norm, namely:

$$|||y||| = \sup(\{|\langle \otimes^n e, y\rangle|; e \in E, \|e\| \leqslant 1\})$$

and clearly $|||y||| \leqslant \|y\|$. Conversely let ζ be a primitive nth root of the unity; for $e_1, \ldots, e_n \in E$ such that $\|e_1\| = \cdots = \|e_n\|$ and $y \in S^n E^\perp$, the algebraic identity

$$\langle e_1 \otimes \cdots \otimes e_n, y\rangle = \left(\frac{1}{n!}\right)^2 \sum_\sigma (\langle \otimes^n (\zeta^{\sigma(1)} e_1 + \cdots + \zeta^{\sigma(n)} e_n), y\rangle \\ - \langle \otimes^n e_1 + \cdots + \otimes^n e_n, y\rangle)$$

shows that $|||y||| \leqslant (n^n + n)/n! \|y\|$, and the two norms are (badly) equivalent. Using (CBS1) as a definition, $|||\,.\,|||$ induces in turn another norm $|||\,.\,|||$ on $T^n E$ and for $x \in T^n E$ $\|x\| \leqslant |||x||| \leqslant (n^n + n)/n! \|x\|$. $T^n E$ and $S^n E$ equipped with their respective norms $|||.|||$ still form a CBS.

3.2. Exponentials

Definition 7. Let E be a CBS; we define the CBS $!E = (!E, ?E^\perp, \langle .,.\rangle)$ as follows:

- $?E$ is the set of all bounded analytical maps φ from the open unit ball of E into $\mathbb{C}$, with coefficients in $E^\perp$. By this we mean that one can find $\varphi_0 \in \mathbb{C}\,(= S^0 E^\perp), \varphi_1 \in E^\perp\ (= S^1 E^\perp), \ldots, \varphi_n \in S^n E^\perp, \ldots$ such that for any $e \in E$ with $\|e\| < 1$ we have

  $$\varphi(e) = \sum_n \varphi_n(\otimes^n e)$$

  (we mean that the complex series converges); boundedness enables one to define the l^∞-norm

  $$\|\varphi\| = \sup\{|\varphi(e)|; \|e\| < 1\}.$$

- $!E$ is defined as the closure in the dual of $?E^\perp$ of the linear span of the forms $!e$, defined for e in the open unit ball of E by

  $$(!e)(\varphi) = \varphi(e).$$

- Since $!E$ is defined as a subspace of $?E^{\perp\prime}$, there is a canonical bilinear form from $!E, ?E^\perp$ into $\mathbb{C}$.

Condition (CBS1) is satisfied by definition. Conversely, observe that $\|!e\| = 1$ for any e in the unit ball of E, from which (CBS2) is easily obtained.

The standard way to cope with analytical functions is to reduce to the familiar complex case: typically if $e, x \in E$, $x \neq 0, \|e\| < 1$, the function $\psi(z) = \varphi(e + x.z)$ is defined for $|z| < (1 - \|e\|)/\|x\|$ and analytical in the usual sense, and we can apply the familiar methods, essentially the Cauchy integral. For instance, let $e=0, \|x\|<1$; the Cauchy estimates (see also Proposition 2) yield $|\varphi_n(\otimes^n x)| \leqslant \|\varphi\|$, hence $|||\varphi_n||| \leqslant \|\varphi\|$, which shows that the monomials φ_n, seen as elements of $?E^\perp$ have a smaller norm than φ.

Proposition 1. *Let $\varphi \in ?E^\perp$ and $e \in E$, $\|e\|<1$; then φ has a derivative $\varphi'_e \in E^\perp$, and $\|\varphi'_e\| \leqslant \|\varphi\|/(1-\|e\|)$.*

Proof. Let us first assume that φ is a monomial φ_n, then it is clearly derivable: $\varphi'_e(x) = n.\varphi_n(\otimes^{n-1} e \otimes x) = 1/n \sum \zeta^i \varphi_n(\zeta^i.e + x)$, and we get a bound $(\|e\| + \|x\|)^n.\|\varphi\|$ on the norm of the derivative $\varphi'_e(x)$. In the general case, a candidate for the derivative of φ at point e is the series $\varphi'_e(x) = \sum n.\varphi_n(\otimes^{n-1} e \otimes x)$; the term $n.\varphi_n(\otimes^{n-1} e \otimes x)$ is the derivative of a monomial whose norm does not exceed that of φ, hence is bounded in norm by $(\|e\|+\|x\|)^n.\|\varphi\|$, and since our series is linear in x we can decide to choose x small enough so that our series is majorized by a geometrical series. This proves the existence of the derivative and that $\varphi'_e \in E^\perp$, as a limit of elements of $E^\perp$. We can also estimate the norm of $\varphi'_e(x)$ by means of the Cauchy integral: let $0<\varepsilon<1-\|e\|$ and let $x \in E, \|x\|=1$, then

$$\varphi'_e(x) = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} \varphi(e + x.\varepsilon.\mathrm{e}^{\mathrm{i}t}).\mathrm{e}^{-\mathrm{i}t}\,\mathrm{d}t,$$

hence $\|\varphi'_e(x)\| \leqslant \|\varphi\|/\varepsilon$, which proves the last claim. □

Proposition 2. *Let us fix $m \in \mathbb{N}, f \in E$ with $0 \leqslant \|f\| < 1$; then there is a point $!_m f \in !E$ such that, for any $\varphi \in ?E^\perp$ with coefficients φ_n the following holds*:

$$\varphi_m(f) = \langle !_m f, \varphi \rangle.$$

Proof. The Cauchy integral

$$\varphi_m(f) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(f.\mathrm{e}^{\mathrm{i}t}).\mathrm{e}^{-\mathrm{i}mt}\,\mathrm{d}t$$

is indeed a Riemann integral, which exploits the uniform continuity of the function $h(t) = (1/2\pi)\varphi(f.\mathrm{e}^{\mathrm{i}t}).\mathrm{e}^{-\mathrm{i}mt}$. A Riemann integral is obtained as the limit of finite barycenters $a_n = 2^{-n} \sum_k h(t.k.2^{-n})$. A consequence of Proposition 1 is that $|h(t) - h(t')| \leqslant 1/2\pi\|\varphi\|(1/(1-\|f\|)+m)|t-t'|$, and from this $|a_n - a_{n'}| \leqslant 2^{-n}\|\varphi\|(1/(1-\|f\|)+m)$ for $n' \geqslant n$, and $\varphi_m(f)$ is the limit of the Cauchy sequence (a_n). Now define $A_n = (1/2\pi)2^{-n}\sum_k \mathrm{e}^{-\mathrm{i}mk2^{-n}} !(f.\mathrm{e}^{\mathrm{i}k2^{-n}})$, so that $a_n = \langle A_n, \varphi \rangle$. We have indeed proven that (A_n) is a Cauchy sequence in $!E$ and we can define $!_m f$ as its limit. □

Our first goal is to give a direct description of intuitionistic implication, which is defined as usual by: $E \Rightarrow F = !E \multimap F = ?E^\perp \parr F$.

Theorem 3. *$E \Rightarrow F$ consists (in fact: is isomorphic to the set) of all bounded analytical functions from the unit ball of E to F defined by a power series*

$$\varphi(e) = \sum \varphi_n(e)$$

with "coefficients" φ_n in $T^nE \multimap F$; furthermore, the norm of $E \Rightarrow F$ is the l^∞-norm

$$\|\varphi\| = \sup\{\|\varphi(e)\|; \|e\| < 1\}.$$

Proof.

- Let φ be defined by such a series; if $f' \in F^\perp$, then $\langle \varphi(e), f' \rangle = \sum \langle \varphi_n(e), f' \rangle$ defines a function $\varphi^{f'}$ which clearly belongs to $?E^\perp$, hence φ induces a map from $F^\perp$ to $?E^\perp$; this map is bounded, since $\|\varphi^{f'}\| \leqslant \|\varphi\|.\|f'\|$. When x is a linear combination of vectors $!e_i$ then we can find $y = T\varphi(x) \in F$ such that $\langle y, f' \rangle = \varphi^{f'}(x)$ for all $f' \in F^\perp$; but $|\varphi^{f'}(x)| \leqslant \|\varphi\|.\|f'\|.\|x\|$. Therefore $\|T\varphi(x)\| \leqslant \|\varphi\|.\|x\|$ and the function $T\varphi$ extends into a map from $!E$ into F. We have therefore shown that φ can be seen as an element of $E \Rightarrow F$, with a smaller norm. Indeed the two norms are easily shown to be equal.
- Conversely, take a function $\varphi \in E \Rightarrow F$. Given $f' \in F^\perp$, consider $\varphi^\perp(f') \in ?E^\perp$; this function is defined by means of a series $\sum \varphi_n^{f'}(e)$, and it is immediate (from the unicity of the expansion which follows from – say – Proposition 2) that the coefficient $\varphi_n^{f'}$ is a linear function of f'; moreover, it is bounded, in virtue of the Cauchy majorization which yields, for $\|e\| < 1$: $\|\varphi_n^{f'}(e)\| \leqslant \|\varphi^{f'}\| \leqslant \|\varphi\|.\|f'\|$, i.e. one can write $\varphi_n^{f'} = \varphi_n(f')$ for an appropriate bounded map φ_n. It remains to show that this map belongs to $T^nE \multimap F$, and the only non-trivial thing to check is that $\varphi_n(e) \in F$ when $e \in E, \|e\| \leqslant 1$; but by Proposition 2 $\varphi_n^{f'}(e) = \langle !_n e, \varphi^{f'} \rangle$, which entails $\varphi_n(e) = \varphi(!_n e)$ and we are done. $\square$

The next theorem is the most important feature of our construction, and justifies the name "exponential" given to ! "Of course" and ? "Why not": ! transforms & (additive conjunction) into $\otimes$ (multiplicative conjunction).

Theorem 4.

$$!(E \,\&\, F) \simeq !E \otimes !F.$$

Proof. We shall prove the dual form

$$?(E^\perp \oplus F^\perp) \simeq ?E^\perp \parr ?F^\perp.$$

Observe that this is a particular case of the more general

$$(E \,\&\, F) \Rightarrow G \simeq E \Rightarrow (F \Rightarrow G)$$

(take $G = \mathbb{C}$), which in turn reduces to the theorem. The isomorphism is the expected one: a function φ defined on the unit ball of $E \,\&\, F$ can be seen as a function Φ sending the unit ball of E into a function defined on the unit ball of F, and conversely. The correspondence is expressed by the formula

$$\Phi(e)(f) = \varphi(e \,\&\, f).$$

What is obvious about this transformation is that it is norm-preserving: this is because both & and $\Rightarrow$ are handled in terms of l^∞-norms

$$\sup\{\sup\{|\Phi(e)(f)|;\ \|f\|<1\};\ \|e\|<1\} = \sup\{|\varphi(e \,\&\, f)|;\ \|e \,\&\, f\|<1\}.$$

The operations are clearly reciprocal, but one must show that they range into the right spaces. Going to the essential, one is reduced to showing an equality of power series

$$\sum_{nm} \varphi_{nm}(e,f) = \sum_{p} \sum_{n+m=p} \varphi_{nm}(e,f).$$

The equality holds because we are in an open ball: choose $\lambda > 1$ with $\lambda\|e \,\&\, f\| < 1$ and let $a_{nm} = \varphi_{nm}(e,f)$; we must show the equality

$$\sum_{nm} a_{nm} = \sum_{p} \sum_{n+m=p} a_{nm}$$

in two cases:

- Under the hypothesis of the convergence of

$$\sum_{nm} \lambda^{n+m} a_{nm}.$$

- Under the hypothesis of the convergence of

$$\sum_{p} \lambda^{p} \sum_{n+m=p} a_{nm}.$$

Both sides follow from standard manipulations on geometric series. $\square$

In the same spirit, observe that $!0 \simeq \mathbb{C}$.

3.3. Comonoids

Definition 8. A (cocommutative) comonoid consists of the following data

- A CBS C,
- A morphism $\mathbf{w} \in C \multimap \mathbb{C}$, with $\|\mathbf{w}\| \leqslant 1$ (counit, or weakening),
- A morphism $\mathbf{c}, \in C \multimap C \otimes C$, with $\|\mathbf{c}\| \leqslant 1$ (comultiplication or contraction),

which enable one to define $\mathbf{c}_i \in C \multimap \otimes^n C$ by $\mathbf{c}_0 = \mathbf{w}, \mathbf{c}_1 = Id, \mathbf{c}_2 = \mathbf{c}, \mathbf{c}_{n+1} = (\mathbf{c} \otimes \mathbf{c}_n)\mathbf{c}$. We require that the $\mathbf{c}_n$ satisfy a coherence property: $(\mathbf{c}_n \otimes \mathbf{c}_m)\mathbf{c} = \mathbf{c}_{n+m}$, which is a dualized form of neutrality and associativity. Furthermore $\sigma\mathbf{c}_n = \mathbf{c}_n$, for any permutation σ: this is cocommutativity.

Co-(neutrality, associativity, commutativity) can be summarized by means of the diagrams:

$$\begin{tikzcd} C \arrow[rr, "\mathbf{c}"] \arrow[dr, "\mathbf{w}"'] & & C \otimes C \arrow[dl, "\mathbf{w} \otimes \mathbf{w}"] \\ & \mathbb{C} & \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\mathbf{c}"] \arrow[d, "\mathbf{c}"'] & C \otimes C \arrow[d, "C \otimes \mathbf{c}"] \\ C \otimes C \arrow[r, "\mathbf{c} \otimes C"'] & C \otimes C \otimes C \end{tikzcd}$$

and

$$\begin{tikzcd} C \arrow[rr, "\mathbf{c}"] \arrow[dr, "\mathbf{c}"'] & & C \otimes C \arrow[dl, "\sigma"] \\ & C \otimes C & \end{tikzcd}$$

where σ stands for the "flip" between two copies of C and C stands for the identity map of C; we neglected the fact that $\otimes$ is not literally associative. $!E$ is naturally endowed with a structure of (cocommutative) comonoid:

- We can define a map $\mathbf{w} \in\, !E \multimap \mathbb{C}$, corresponding to the analytical function constantly equal to 1.
- There is a unique map $\mathbf{c} \in\, !E \multimap !E \otimes !E$ such that $\mathbf{c}(!e) = !e \otimes !e$; in fact one can see $\mathbf{c}^{\perp}$ as the map taking a binary analytical function $\varphi(e,f)$ into the unary function $\varphi(e,e)$. From this $\|\mathbf{c}\| \leqslant 1$.

The map $\mathbf{c}_n \in\, !E \multimap \otimes^n !E$ is uniquely defined by $\mathbf{c}_n(!e) = \otimes^n !e$. From this, co-associativity, neutrality, commutativity are immediate.

Definition 9. Let C, D be comonoids; a morphism $\varphi \in C \multimap D$ is said to be a morphism of comonoids when the following hold:

- $\mathbf{w}\varphi = \mathbf{w}$ (preservation of counit)
- $\mathbf{c}\varphi = (\varphi \otimes \varphi)\mathbf{c}$ (preservation of comultiplication)

i.e. the commutation of the diagrams:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\mathbf{c}_n"'] & D \arrow[d, "\mathbf{c}_n"] \\ \otimes^n C \arrow[r, "\otimes^n \varphi"'] & \otimes^n D \end{tikzcd}$$

There is an important map connected with $!E$, namely $\mathbf{d} \in !E \multimap E$ (*dereliction*), the only map such that $\mathbf{d}(!e) = e$; dually $\mathbf{d}^{\perp}$ takes a vector $e' \in E^{\perp}$ into the corresponding analytical map $e \rightsquigarrow \langle e, e' \rangle$. The importance of dereliction is stressed by the

Theorem 5. *Let C be a comonoid let E be a CBS and let $\varphi \in C \multimap E$, with $\|\varphi\| < 1/e$; then there exists a unique morphism of comonoids $!\varphi \in C \multimap !E$ such that $\varphi = \mathbf{d}!\varphi$:*

$$\begin{tikzcd} C \arrow[rr, "!\varphi"] \arrow[dr, "\varphi"'] & & !E \arrow[dl, "\mathbf{d}"] \\ & E & \end{tikzcd}$$

Proof. If $x \in C$, define $!\varphi(x) = \sum_n (\otimes^n \varphi)(c_n(x))$; the nth term of the series officially belongs to $\otimes^n E$, but an immediate inspection shows that it is indeed in its symmetrical part $T^n E$ which (when equipped with the norm $|||\,.\,|||$) is a subspace of $!E$. Everything is almost trivial, but convergence.

Let $u_n = |||(\otimes^n \varphi)(\mathbf{c}_n(x))|||$. Then

$$u_n \leqslant \frac{n^n + n}{n!} ||(\otimes^n \varphi)(\mathbf{c}_n(x))|| \leqslant \frac{n^n + n}{n!} ||\varphi||^n . ||\mathbf{c}_n(x)|| \leqslant \frac{n^n + n}{n!} ||\varphi||^n . ||x||$$

The result follows from the Stirling formula; slightly more directly, with $v_n = (n^n + n)/n! ||\varphi||^n . ||x||$, of the estimate of the limit of $(v_{n+1})/v_n$: $(v_{n+1})/v_n \sim ||\varphi||.(1 + (1/n))^n$ tends to $||\varphi||.e < 1$. $\square$

The theorem does not quite establish $!E$ as the solution of a universal problem, due to this restriction on the norm of the input φ. So let us change the definition of a comonoid: due to cocommutativity, the map $\mathbf{c}_n$ actually ranges into $T^n C$, hence we can consider the norm $|||\mathbf{c}_n||| = \sup\{|||\mathbf{c}_n(c)|||; ||c|| \leqslant 1\}$.

Definition 10. A *strong* comonoid is defined as in Definition 8, except that we now require:

$$|||\mathbf{c}_n||| \leqslant 1.$$

Proposition 3. *$!E$ is a strong comonoid; furthermore $(!E, \mathbf{d})$ is the solution to the problem of Theorem 5, but with the more liberal hypothesis $||\varphi|| < 1$.*

Proof. First $!E$ is a strong comonoid: this is because $\mathbf{c}_n$ is indeed an analytical map from E to $T^n!E$ defined by $\mathbf{c}_n(e) = \otimes^n!e$, and its norm is therefore 1. The second-half of the statement follows from an inspection of the proof of Theorem 5:

$$u_n \leqslant ||\varphi||^n.|||\mathbf{c}_n(x)||| \leqslant ||\varphi||^n.|||\mathbf{c}_n|||.||x|| \leqslant ||\varphi||^n.||x|| \qquad \Box$$

With respect to strong comonoids, $!E$ is "almost" the solution to a universal problem; there is a small mismatch, namely that $||\mathbf{d}|| = 1$, whereas the φ to which the property applies cannot reach the norm 1.

4. Implementation of first-order linear logic

In the sequel we ignore quantification: the first-order case can basically be handled on the model of $\oplus$ and &, whereas the more essential second-order case is out of reach for the moment.

4.1. Formulas and sequents

In order to present the calculus, we shall adopt the following notational simplification: formulas are written from literals p, q, r, $p^\perp$, $q^\perp$, $r^\perp$, etc., and constants $\mathbf{1}$, $\perp$, $\top$, $\mathbf{0}$ by means of the connectives $\otimes$, $⅋$, &, $\oplus$ (binary), !, ? (unary). Negation is *defined* by De Morgan equations, and linear implication is also a defined connective:

$$\begin{array}{rclrcl}
\mathbf{1}^\perp & := & \perp & \perp^\perp & := & \mathbf{1} \\
\top^\perp & := & \mathbf{0} & \mathbf{0}^\perp & := & \top \\
(p)^\perp & := & p^\perp & (p^\perp)^\perp & := & p \\
(A \otimes B)^\perp & := & A^\perp ⅋ B^\perp & (A ⅋ B)^\perp & := & A^\perp \otimes B^\perp \\
(A \& B)^\perp & := & A^\perp \oplus B^\perp & (A \oplus B)^\perp & := & A^\perp \& B^\perp \\
(!A)^\perp & := & ?A^\perp & (?A)^\perp & := & !A^\perp
\end{array}$$
$$A \multimap B := A^\perp ⅋ B.$$

The connectives $\otimes$, $⅋$, $\multimap$, together with the neutral elements $\mathbf{1}$ (w.r.t. $\otimes$) and $\perp$ (w.r.t. $⅋$) are called *multiplicatives*; the connectives & and $\oplus$, together with the neutral elements $\top$ (w.r.t. &) and $\mathbf{0}$ (w.r.t $\oplus$) are called *additives*; the connectives ! and ? are called *exponentials*. The notation has been chosen for its mnemonic virtues: we can remember from the notation that $\otimes$ is multiplicative and conjunctive, with neutral $\mathbf{1}$, $\oplus$ is additive and disjunctive, with neutral $\mathbf{0}$, that $⅋$ is disjunctive with neutral $\perp$, and that & is conjunctive with neutral $\top$; the distributivity of $\otimes$ over $\oplus$ is also suggested by our notation.

4.2. Sequents

Sequent calculus is the traditional tool of proof-theory; we are basically using a variant of this calculus that we introduced in the paper [6] and which induces a great syntactical flexibility.

Definition 11.
- A *discharged* formula is an expression $[A]$, where A is a formula;
- A *sequent* is an expression $\vdash \mathbf{A}_1, \ldots, \mathbf{A}_n$, where $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are either discharged formulas or formulas.

Remark. Formulas will be implemented by CBS, following the definitions of the previous sections; then sequents will be also implemented by CBS, since commas and [.] are just another way to speak of "par" and "why not":
- ▶ A discharged formula $[A]$ is hypocrisy for $?A$;
- ▶ If $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are hypocrisy for formulas $B_1, \ldots, B_n$, then the sequent $\vdash \mathbf{A}_1, \ldots, \mathbf{A}_n$ is hypocrisy for the formula $B_1 \mathbin{⅋} \cdots \mathbin{⅋} B_n$.

In sequent calculus we shall (see below) prove sequents, and each proof of $\vdash \Gamma$ will indeed be implemented as a vector γ of the corresponding CBS.

4.3. Weighted sequent calculus

Identity/Negation:

$$\frac{}{\vdash_1 A, A^\perp}\ (identity) \qquad \frac{\vdash_\lambda \Gamma, A \quad \vdash_\mu A^\perp, \Delta}{\vdash_{\lambda\mu} \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash_{1-0} [\Gamma], A \quad \vdash_\lambda [A^\perp], \Delta}{\vdash_\lambda [\Gamma], \Delta}\ ([cut])$$

Structure:

$$\frac{\vdash_\lambda \Gamma}{\vdash_\lambda \sigma(\Gamma)}\ (exchange) \qquad \frac{\vdash_\lambda \Gamma}{\vdash_\lambda \Gamma, [A]}\ (weakening)$$

$$\frac{\vdash_\lambda \Gamma, A}{\vdash_\lambda \Gamma, [A]}\ (dereliction) \qquad \frac{\vdash_\lambda \Gamma, [A], [A]}{\vdash_\lambda \Gamma, [A]}\ (contraction)$$

$$\frac{\vdash_\lambda \Gamma}{\vdash_{\lambda.|\mu|} \Gamma}\ (scalar\colon \mu \neq 0) \qquad \frac{\vdash_\lambda \Gamma}{\vdash_{\lambda'} \Gamma}\ (waste\colon \lambda' > \lambda),$$

$$\frac{\vdash_\lambda \Gamma, A, B}{\vdash_{\lambda-0} \Gamma, A, B}\ (open)$$

Logic:

$$\frac{}{\vdash_\lambda \mathbf{1}} \quad (\textit{one}\colon \lambda > 1) \qquad \frac{\vdash_\lambda \Gamma}{\vdash_\lambda \Gamma, \bot} \quad (\textit{false})$$

$$\frac{}{\vdash_\lambda \Gamma, \top} \quad (\textit{true}) \qquad \frac{\vdash_\lambda \Gamma, A, B}{\vdash_{\lambda+0} \Gamma, A \parr B} \quad (\textit{par})$$

$$\frac{\vdash_\lambda \Gamma, A \quad \vdash_\mu B, \Delta}{\vdash_{\lambda\mu} \Gamma, A \otimes B, \Delta} \quad (\textit{times}) \qquad \frac{\vdash_\lambda \Gamma, A}{\vdash_\lambda \Gamma, A \oplus B} \quad (\textit{left plus})$$

$$\frac{\vdash_\lambda \Gamma, A \quad \vdash_\lambda \Gamma, B}{\vdash_\lambda \Gamma, A \& B} \quad (\textit{with}) \qquad \frac{\vdash_\lambda \Gamma, B}{\vdash_\lambda \Gamma, A \oplus B} \quad (\textit{right plus})$$

$$\frac{\vdash_{1-0} [\Gamma], A}{\vdash_1 [\Gamma], !A]} \quad (\textit{of course}) \qquad \frac{\vdash_\lambda \Gamma, [A]}{\vdash_\lambda \Gamma, ?A} \quad (\textit{why not})$$

This formulation of linear sequent calculus uses indices (weights) λ, μ that we shall soon explain. If we just ignore them, we are left with three useless rules: “scalar”, “open” and “waste”.

4.4. Weighted sequents

Definition 12. A *weight* is an interval $[0, \alpha[$ (notation: $\alpha - 0$) or an interval $[0, \alpha]$ (notation: α); in both cases the real α is strictly positive. Operations on intervals are symbolized as follows:

- We extend usual product by: $\alpha(\beta - 0) = (\alpha - 0)\beta = (\alpha - 0)(\beta - 0) = \alpha\beta - 0$; this operation represents the pointwise product of intervals.
- We extend “-0” by $(\alpha - 0) - 0 = \alpha - 0$; this operation represents the interior of an interval.
- We define $\alpha + 0 = (\alpha - 0) + 0 = \alpha$; this operation represents the closure of an interval.
- We extend the order relation by: $\alpha - 0 < \beta \Leftrightarrow \alpha \leqslant \beta$
  $\alpha - 0 < \beta - 0 \Leftrightarrow \alpha < \beta - 0 \Leftrightarrow \alpha < \beta$; this relation represents strict inclusion of intervals.

Definition 13. A *weighted sequent* is an expression $\vdash_\lambda \Gamma$, where λ is a weight.

Weighted sequent calculus has already been written above. We must now explain the precise meaning of the weights. A proof of $\vdash_\lambda \Gamma$ will be implemented by a vector γ in the appropriate CBS and λ will be a comment about the “size” of γ. More precisely:

- If $\lambda = \alpha + 0$, i.e. if λ denotes a closed interval, we just mean $||\gamma|| \leqslant \alpha$;
- If $\lambda = \alpha - 0$, the condition means the following: select any element A (resp. $[A]$) of Γ, and let $\Delta, [\Delta']$ be the remaining elements of Γ (implemented as spaces). Then γ induces an analytical function φ defined on the open unit ball of the “with” of $\Delta^\perp, \Delta'^\perp$ (φ is multilinear w.r.t. $\Delta^\perp$, hence analytical on its “with”), with values in A (resp. $?A$); we require that φ ranges into the *open* ball of radius α. When Γ contains

at least two non-discharged formulas, the condition collapses to $||\gamma|| \leqslant \alpha$; when Γ has only one element, the condition collapses to $||\gamma|| < \alpha$.

The rules are just (but for the new "scalar", "open" and "waste" rules) decorations of the familiar rules for sequents. If we try to decorate usual sequent calculus proofs with scalars λ, then the obvious candidate is $\lambda = 1$; but one of the rules is problematic, namely "of course", since it requires a premise of weight $1 - 0$ to produce a conclusion of weight 1. This is why we added "scalar" which enables one to change (usually: lower) the weight; but this is not as stupid as you may think, since this rule will be interpreted by a scalar multiplication by μ. "Waste" is a convenient minor rule which enables one to get some flexibility; it is by no means the converse of "scalar", since it is just another estimate on the weight of the same vector. Finally "open" is just the remark that, in presence of two undischarged formulas, the distinction between α and $\alpha - 0$ vanishes.

4.5. Implementation of proofs

Assume that the atoms $p, q, r, \ldots$ are implemented by CBS $P, Q, R, \ldots$; then every formula is immediately implemented as a CBS. (Both multiplicative neutrals $\mathbf{1}, \perp$ are implemented by $\mathbb{C}$, and both additive neutrals $\top, \mathbf{0}$ are implemented by 0). A discharged formula $[A]$ is interpreted as the "why not" of A, and a sequent Γ is implemented as the "par" of its elements.

It remains to implement a proof of $\vdash_\lambda \Gamma$ by a vector of the space interpreting Γ and check that the weight constraints explained in the previous section are satisfied. This is more or less obvious since our system is nothing but a complicated way to speak of the basic constructions of Sections 2 and 3. For instance the Identity axiom is the bilinear form, the two cut-rules are composition with a linear or with an analytical map, and our requirements on λ ensure that the domains/codomains match. Exchange is permutation, weakening and contraction are fake dependency and diagonalization, both in the analytical case, and dereliction is the observation that a linear map is analytical. Scalar is scalar multiplication, whereas waste does not affect the vector. Multiplicative rules basically state the universal property of the tensor product; additive rules are another way to state the universal property of the direct sum. The most important rule is "of course", which can be described as follows: take an n-ary analytical function φ from the unit ball of the "with" of $\Gamma^\perp$ into the *open* unit ball of A, and get an analytical function with values into $!A$. It is enough to define $(!\varphi)(\gamma) = !(\varphi(\gamma))$. The existence of the solution cannot be justified by Theorem 5 or Proposition 3, whose hypotheses are too drastic. But one can explicitly write the power series expansion of φ and that's it by Theorem 3. Another way to see the rule is composition of analytical maps $\psi\varphi$, corresponding to the rule [cut]: when $\psi \in ?A^\perp$, then ψ is indeed an analytical function from the unit ball of A into $\mathbb{C}$, hence $\langle !\varphi(\gamma), \psi \rangle = \psi(\varphi(\gamma))$.

A subtle points: the "par" rule changes $\alpha - 0$ into α: this is necessary in case Γ is fully discharged, typically when Γ is empty; when Γ is not fully discharged, an additional use of "open" enables one keep the weight $\alpha - 0$.

Sequent calculus is organized around the cut-rule, and its main property is cut-elimination: a proof using the cut-rule can be mechanically transformed into a normal proof, without cuts. The main property of our interpretation is that it is invariant under cut-elimination.

Theorem 6. *If a proof Π of $\vdash_\lambda \Gamma$ normalizes into Π', then Π and Π' are implemented by the same vector.*

Proof. The proof is trivial and very long; it resembles one hundred of similar results, and we content ourselves with side comments.
- One needs to define the cut-elimination algorithm; this is straightforward, but very long. The new rules, like "scalar" are not problematic at all, provided one keeps in mind their intuitive meaning.
- Then one must check the invariance of the interpretation under each cut-elimination step. The essential ingredients are to be found in our previous theorems: for instance the main cut-elimination step corresponding to a cut on a multiplicative formula $A \otimes B$ is handled by means of the universal property of the tensor, or if one prefers, the present theorem is just another way to state this universal property.

In fact the best solution would be to develop proof-nets for this modified calculus. Let us just observe that if we take cut-free (i.e. normal) proofs of sequents which do not use the connectives &,!, then they can be (up to some shuffling of rules) written in the following order
1. A usual sequent calculus proof, with all weights set to 1: this is always possible in the absence of !.
2. A single use of "scalar".
3. At most a single use of "waste" or "open".

In simpler terms: normal proofs in the weighted calculus are scalar multiples of normal proofs in the standard calculus. The property fails in the presence of & (several scalars are used) and ! (too badly non-linear).

Appendix A

A.1. The Gustave function

The Gustave function (invented by Berry) is a typical example of non-sequential algorithm. We give here a version adapted to linear logic. Let $A, B, C, A', B', C', A'', B'', C''$ be equal (we name them differently for convenience), and consider $X = (A \,\&\, (B \oplus C)) \otimes (A' \,\&\, B' \oplus C')) \otimes (A'' \,\&\, (B'' \oplus C''))$; then we define a function pointwise from X into $A \otimes A \otimes A$ (with obvious notations: b' stands for an element of B', etc.):

$$\gamma(a \otimes b' \otimes c'') = a \otimes b' \otimes c''$$

$$\gamma(b \otimes c' \otimes a'') = b \otimes c' \otimes a''$$

$$\gamma(c \otimes a' \otimes b'') = c \otimes a' \otimes b''$$

$$\gamma(b \otimes b' \otimes b'') = b \otimes b' \otimes b''$$

$$\gamma(c \otimes c' \otimes c'') = c \otimes c' \otimes c''$$

Usual denotational semantics, including coherent spaces accept this function; however Ehrhard [4] has been able to introduce *hypercoherences*, a beautiful generalization of coherent spaces, in which Gustave is not accepted as a clique. The question is therefore whether or not CBS accept this function. In other terms, let us implement $A, B, \ldots$, by a non-zero space E and let us compute $||\gamma||$. Since γ is defined on a ternary tensor product, the norm of γ is equal to its norm as a trilinear map, i.e.

$$||\gamma|| = \sup\{||\gamma((a \,\&\, (b \oplus c)) \otimes ((a' \& (b' \oplus c')) \otimes ((a'' \,\&\, (b'' \oplus c''))||\},$$

the supremum being taken over those $a, b, c \ldots$ such that:

$$||(a \& (b \oplus c)||, ||(a' \& (b' \oplus c')||, ||(a'' \& (b'' \oplus c'')|| \leqslant 1.$$

Let $\alpha = ||a||$, $\alpha' = ||a'||, \ldots$ then $||a|| \& (b \oplus c)|| = \max(\alpha, \beta + \gamma)$, etc. We are led to majorize $\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' + \beta\beta'\beta'' + \gamma\gamma'\gamma''$; we can assume w.l.o.g. that $\alpha = \beta + \gamma = 1$, etc., and then our expression is shown to equal 1, which proves that $||\gamma|| = 1$.

Unfortunately the answer is positive, which shows that certain important features of denotational semantics have not been caught by CBS. Does this mean that the notion must be refined?

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